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LU-Factorization of Order Bounded Operators on Banach Sequence Spaces

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The topic of LU-factorization of operators on Banach spaces has attracted a great deal of attention in recent years, particularly from workers in approximation theory. From their point of view, this topic may be considered as a problem in "infinite dimensional numerical analysis." Thus far, factorization theorems have been obtained for invertible, totally positive operators on l_∞ [4] and for Toeplitz totally positive matrices [5]. These theorems have proven useful in connection with spline interpolation problems [6] and certain time invariant linear systems [9]. It has also been established that invertible, diagonally dominant operators on l_1 have LU-factorizations. Thus, work in this area for the most part has centered on matrix operators on l_∞ and l_1 (and, to a lesser extent, l_2 [6]).

There are good reasons for this. For matrix operators on l_∞ and l_1, the usual operator norm can be calculated easily from the entries of the matrix. Moreover, the upper and lower triangular parts of such operators are also bounded operators on such spaces. For the other l_p's this is no longer true. For example, the Hilbert–Toeplitz operator on l_2 that is represented by the matrix

$$\begin{pmatrix} 0 & -1 & -\frac{1}{2} & -\frac{1}{3} & \\ 1 & 0 & -1 & -\frac{1}{2} & \\ \frac{1}{2} & 1 & 0 & -1 & \\ \frac{1}{3} & \frac{1}{2} & 1 & 0 & \ddots \end{pmatrix}$$

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is a bounded linear operator on l_2 but its lower triangular part is not [8, p. 51]. Nevertheless, it is possible to extend the results on factorization of invertible, totally positive operators, matrix operators on l_∞, and invertible, diagonally dominant operators on l_1 to the other l_p spaces and to c_0. This is the main purpose of this paper. We make this extension by characterizing when a certain class of operators, called order bounded operators, has a strong type of LU-factorization. This class of operators includes totally positive matrix operators on l_∞ and diagonally dominant operators on l_1. We also give an application of LU-factorization to the solution of certain operator equations. It is now time to fix some terminology and notation.

Throughout this paper we will only consider operators on the Banach spaces c_0 and l_p, $1 \leqslant p < +\infty$. We will refer to these spaces as classical Banach sequence spaces. Operators on such spaces can naturally be represented by means of infinite matrices. Let X denote a classical Banach sequence space. For each subset L of the positive integers we define a norm 1 projection operator P_L on X by $P_L x = \sum_{i \in L} x_i e_i$ for all $x = (x_i) \in X$. (Here e_i denotes a member of the usual vector basis of X.) In case $L = \{1, 2, ..., n\}$ we denote P_L by P_n. An operator T on X is said to be *upper* (*respectively lower*) *triangular* if $P_n T P_n = T P_n$ (respectively $P_n T P_n = P_n T$) for all n. An operator T is said to be *diagonal* if $T P_n = P_n T$ for all n. We say that an operator T is *unit upper* (*lower*) *triangular* if it is upper (lower) triangular and its diagonal entries are all 1's. An operator T is said to have an *LU-factorization* (relative to the usual basis e_i) if there exist invertible operators L and U so that $T = LU$ and the operators L, L^{-1} are unit lower triangular while U, U^{-1} are upper triangular.

We recall [7, p. 178] that a finite $m \times m$ matrix T has an LU-factorization relative to the usual basis $\{e_1, ..., e_m\}$ if and only if for each $n = 1, 2 ... m$ the compression $T_n = P_n T P_n$ is invertible as an operator on the span of $\{e_1, ..., e_n\}$. Moreover, the upper triangular matrix U can be obtained by Gauss elimination. From this point of view the results on LU-factorization of operators on infinite-dimensional Banach spaces may be thought of as a partial answer to the question of when Gauss elimination on infinite matrices gives rise to bounded upper triangular operators. The reason why the answer is partial is that the above mentioned equivalence does not hold in the infinite dimensional case.

Barkar and Gohberg [1] have shown that if T is an operator on a classical Banach sequence space and if T has an LU-factorization, then T and all its compressions T_n are invertible. But the converse is not true, as the following example of R. R. Smith illustrates.

EXAMPLE. Consider the operator T on l_2 which has 2×2 blocks of the form

$$\begin{pmatrix} \frac{1}{2+4(j-1)} & 1 \\ 1 & \frac{1}{4j} \end{pmatrix}$$

$j = 1, 2...$ arranged along the main diagonal. Then it is easy to see that T can be written as the sum of a diagonal operator of norm $\frac{1}{2}$ and a unitary operator which has the 2×2 blocks $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ arranged along the main diagonal. Thus T is invertible. The compressions T_n of the operator T are also invertible (relative to the subspace $P_n(l_2)$), since if n is even, T_n consists of invertible 2×2 blocks while if n is odd, T_n consists of invertible 2×2 blocks and a nonzero (n, n) entry. But, if n is odd then $T(e_n) = e_n/2n$, so $\|T_n\| \geqslant 2n$. Consequently, $\sup_n \|T_n^{-1}\| = +\infty$, so T cannot have an LU-factorization [1, Theorem 2]. In light of this example and the existing factorization results it is natural to conjecture that if an operator T on a classical Banach sequence space is invertible and has invertible compressions T_n satisfying $\sup_n \|T_n^{-1}\| < +\infty$, then the operator T has an LU-factorization. We have been unable to show this. What we can show is that a stronger condition on the compressions is equivalent to the existence of a stronger type of LU-factorization. This result includes most of the known results and permits extensions of them to other l_p spaces. In this process, the notion of an order bounded operator plays a central role. If $T = (t_{ij})$ is an operator on a classical Banach sequence space then T is said to be *order bounded* if $|T| = (|t_{ij}|)$ is also a bounded linear operator on the same space. (The term absolutely bounded is also used [8, p. 50].) We note that every bounded T operator on c_0 or l_1 is order bounded. A set A of order bounded operators on a classical Banach sequence space is said to be order bounded if $\sup_{T \in A} |T| = (\sup_{T \in A} |t_{ij}|)$ is also a bounded linear operator on the same space. Before stating the main theorem, we require an elementary result that could also be obtained using results of [1].

PROPOSITION 1. *Let T be an $n \times n$ matrix. If T has an LU-factorization then*

$$L^{-1}(i, j) = -\sum_{k=1}^{i-1} T_{i-1}^{-1}(k, j)\, T(i, k) \qquad \textit{for} \quad i > j$$

and

$$U^{-1}(i, j) = T_j^{-1}(i, j) \qquad \textit{for} \quad i < j.$$

Proof. Let $\vec{r}_i$ denote the ith row of L^{-1} and $\vec{t}_j$ the jth column of T. Since $L^{-1}T = U$ and U is upper triangular, it follows that $\vec{r}_i \cdot \vec{t}_j = 0$ for $i > j$. Since $L^{-1}(i, i) = 1$ for all i, we have that

$$\sum_{k=1}^{i-1} L^{-1}(i, k) \langle T(k, 1), \ldots, T(k, i-1) \rangle = -\langle T(i, 1), \ldots, T(i-1) \rangle.$$

Hence

$$\langle L^{-1}(i,1),\ldots,L^{-1}(i,i-1)\rangle T_{i-1} = -\langle T(i,1),\ldots,T(i,i-1)\rangle,$$

and so

$$\langle L^{-1}(i,1),\ldots,L^{-1}(i,i-1)\rangle = -\langle T(i,1),\ldots,T(i,i-1)\rangle T_{i-1}^{-1}.$$

In particular, $L^{-1}(i,j) = -\sum_{k=1}^{i-1} T(i,k)\, T_{i-1}^{-1}(k,j)$ for $j<i$.

Similarly, one can show that $U^{-1}(i,j) = T_j^{-1}(i,j)$ for $i<j$. This completes the proof.

To motivate our main result, we note that if T is an operator whose compressions T_n have LU-factorizations $L_n U_n$, then $L_n = P_n L_{n+1} P_n$ and $U_n = P_n U_{n+1} P_n$ for all n. Consequently, both $L = \mathrm{Lim}_n L_n$ and $U = \lim U_n$ exist formally, where the limits are taken entrywise. These matrices are natural candidates for an LU-factorization for T. The difficulty is that L and U may not represented *bounded* operators. By Proposition 1, we can see that if T is an operator on l_1 with $\sup_n \|T_n^{-1}\| < +\infty$, then $\|L\| \leqslant \sup_n \|L_n\| = \sup_n \|T_n U_n^{-1}\| \leqslant \|T\| \sup_n \|U_n^{-1}\| \leqslant \|T\| \sup_n \|T_n^{-1}\| < +\infty$. To ensure that U is bounded an (apparently) further condition on T_n^{-1} is needed as the next result shows, since bounded operators on l_1 are order bounded.

THEOREM 2. *Let T be an order bounded operator on a classical Banach sequence space X. Then T has an LU-factorization such that L^{-1} and U^{-1} are order bounded if and only if the following conditions hold*:

(i) *For each n, the compression $T_n = P_n T P_n$ is invertible on $P_n(X)$.*

(ii) *The set of inverses $\{T_n^{-1} : n \in N\}$ is order bounded.*

Proof. For the forward implication, we note that if $T = LU$ then $T_n = P_n(LU) P_n = (P_n L P_n)(P_n U P_n)$ and hence T_n is invertible. In fact, $T_n^{-1} = (P_n U^{-1} P_n)(P_n L^{-1} P_n)$ so $T_n^{-1}(i,j) = \sum_{k=1}^{n} U^{-1}(i,k)\, L^{-1}(k,j)$. Hence, if $X = l_p$, $1 \leqslant p < +\infty$, then

$$\begin{aligned}
\|\sup_n |T_n^{-1}|\,\| &= \sup_{\|x\| \leqslant 1} \left(\sum_{i=1}^{+\infty} \left| \sum_{j=1}^{+\infty} \sup_n |T_n^{-1}(i,j)| |x(j)| \right|^p \right)^{1/p} \\
&\leqslant \sup_{\|x\| \leqslant 1} \left(\sum_{i=1}^{+\infty} \left(\sum_{k=1}^{+\infty} |U^{-1}(i,k)| \sum_{j=1}^{+\infty} |L^{-1}(k,j)| |x(j)| \right)^p \right)^{1/p} \\
&\leqslant \sup_{\|x\| \leqslant 1} \|\,|U^{-1}|\,\|\,\|\,|L^{-1}|\,\|\,\|x\| = \|\,|U^{-1}|\,\|\,\|\,|L^{-1}|\,\|.
\end{aligned}$$

With obvious modifications, the same proof works for $X = c_0$.

For the reverse implication, note that the hypothesis implies that each T_n has an LU-factorization [7, p. 178]. We show that the sets $\{L_n^{-1}: n \in N\}$ and $\{U_n^{-1}: n \in N\}$ are order bounded where $T_n = L_n U_n$. If $X = l_p$ then by Proposition 1,

$$\begin{aligned}
\|\sup|L_n^{-1}|\,\| &\leqslant \sup_{\|x\| \leqslant 1} \left(\left(\sum_{i=1}^{+\infty} \left| \sum_{j=1}^{+\infty} \sup_n |L_n^{-1}(i,j)|\,|x(j)| \right|^p \right)^{1/p} + 1 \right. \\
&\leqslant \sup_{\|x\| \leqslant 1} \left(\sum_{i=1}^{+\infty} \left| \sum_{j=1}^{+\infty} \sum_{k=1}^{+\infty} \sup_i |T_{i-1}^{-1}(k,j)|\,|T(i,k)|\,|x(j)| \right|^p \right)^{1/p} + 1 \\
&\leqslant \|\,|T|\,\|\,\|\sup_n |T_n^{-1}|\,\| + 1 < \infty.
\end{aligned}$$

Similarly, $\|\sup_n |U_n^{-1}|\,\| \leqslant \|\sup_n |T_n^{-1}|\,\| < +\infty$ and these results are also true if $X = c_0$. Since $L_n = P_n L_{n+1} P_n$ and $U_n = P_n U_{n+1} P_n$, it follows that $L_n^{-1} = P_n L_{n+1}^{-1} P_n$ and $U_n^{-1} = P_n U_{n+1}^{-1} P_n$. Consequently, for each x in X, the limits $\lim_n L_n x = Lx$, $\lim_n L_n^{-1} x = Vx$, $\lim_n U_n x = Ux$, and $\lim_n U_n^{-1} x = Wx$ exist and define bounded triangular linear operators on X. In fact, it is clear that V and W are order bounded. Now since $x = \lim_n \mathbf{I}_n x = \lim_n L_n L_n^{-1} x = LVx$ and $x = \lim_n \mathbf{I}_n x = \lim_n L_n^{-1} L_n x = VLx$, we have that $V = L^{-1}$. Similarly $W = U^{-1}$. Finally, for each $x \in X$, $LUx = \lim_n \lim_n L_n U_n x = \lim_n T_n x = Tx$ so T has the promised factorization.

Our first corollary deals with totally positive operators on classical Banach sequence spaces. An operator $T = (t_{ij})$ is *totally positive* if for all positive integers $i_1 < i_2 < \cdots < i_n$, $j_1 < j_2 < \cdots < j_n$, $n \geqslant 1$, we have that $\det(t_{i_k j_l}) \geqslant 0$. Obviously, such operators are order bounded. In [4], De Boor, Jia, and Pinkus have shown that invertible, totally positive matrix operators on l_∞ have LU-factorizations. It follows easily from their results that invertible, totally positive operators on c_0 and l_1 have LU-factorizations. Here we extend those results to other l_p spaces. The proof is based on several ideas used extensively in [3] and [4]. We remark (using the language of these papers) that since the operators we consider here are represented by infinite rather than biinfinite matrices the "main diagonal" of an invertible totally positive operator is the "0th" diagonal.

COROLLARY 3. *Let T be an invertible, totally positive operator on a classical Banach sequence space X. Then T has an LU-factorization such that the operators L^{-1} and U^{-1} are order bounded.*

Proof. Let $u \in X$ be a norm 1 element whose coordinates satisfy $(-1)^i u(i) > 0$ for all i. Let I and J denote the index sets for the rows and columns of T. An examination of the proof of Theorem 1 of [3] reveals that for every finite interval $L \subset \mathbf{I}$, there exists a subset $K \subset J$ with equal cardinality such that $T_{L,K} = P_K T P_L$ is invertible and $\|(T_{L,K})^{-1} e_i\| \leqslant \|T_\mu^{-1}\|/|u(i)|$ for all $i \in L$. Consequently, just as in Theorem 1 of [4], there is a sequence of intervals L increasing to I such that $(T_{L,K})^{-1} e_i \to T^{-1} e_i$ coordinate-wise for all i. It follows that any $j \in J$ is eventually in every K and so for L sufficiently large $T_n = P_n T P_n$ is an upper left-hand submatrix of $T_{L,K}$. Hence T_n is invertible by Hadamard's inequality [10, p. 88]. Moreover, by Lemma 1 of [4], $|T_n^{-1}(j, i)| \leqslant T_{K,L}^{-1}(j, i)|$ and $0 \leqslant (-1)^{i+j} T_n^{-1}(j, i) \leqslant (-1)^{i+j} T_{n+1}^{-1}(j, i)$ for all (j, i). Consequently, $y(j, i) = \lim_n T_n^{-1}(j, i)$ exists for all (j, i) and $\| y(\cdot, i)\| \leqslant \|T^{-1}u\|/|u(i)|$. It follows that $(Ty_i)(k) = \lim_n \sum_i T(k, l)\, T_n^{-1}(l, i) = \langle e_i, e_k \rangle$ and so $y_i = T^{-1} e_i$. Since the entries of T^{-1} form a checkerboard pattern of signs [4], it is easy to see that T^{-1} is order bounded. Hence $\sup_n |T_n^{-1}| = |T^{-1}|$ has a finite norm so an application of Theorem 2 will now give the result.

Our next two results deal with diagonally dominant operators. If $T = (t_{ij})$ is an operator on l_1, then T is said to be (column) *diagonally dominant* if and only if $|t_{jj}| \geqslant \sum_{i \neq j} |t_{ij}|$ for all j. Smith and Ward [14] have shown that an invertible, diagonally dominant operator on l_1 has an LU-factorization. From this it is easy to deduce a factorization result for invertible (row) diagonally dominant operators on c_0. To see this, let T be such an operator. Then T^* is an invertible (column) diagonally dominant operator on l_1 and so has an LU-factorization $\hat{L}\hat{U} = T^*$. Now since each row of $\hat{L}$ and $\hat{L}^{-1}$ is an element of c_0, it follows [15, p. 217] that $\hat{L}$ must be the adjoint of an invertible upper triangular operator U on c_0. Since $\hat{U} = \hat{L}^{-1} T^*$, we have that $\hat{U}$ must also be an adjoint of a (necessarily) lower triangular operator L on c_0. This gives a factorization for T of the form $T = LU$ where U, U^{-1} are unit (upper triangular) operators. From this and Theorem 2 it is clear that T must have an LU factorization where L, L^{-1} are unit (lower triangular) operators. There are at least two ways to extend the notion of diagonal dominance to other classical Banach sequence spaces. The first (and most straightforward) is to say that an operator T on a classical Banach sequence space X is *strictly diagonally dominant* if and only if $\| \, |T_d| x\| > \| \, |(T - T_d)| x\|$ for all $x \in X$, where T_d denotes the diagonal part of T.

COROLLARY 4. *Let T be a strictly diagonally dominant, invertible operator on a classical Banach sequence space. Then T has an LU-factorization with L^{-1} and U^{-1} order bounded.*

Proof. Since T is strictly diagonally dominant and bounded below, it follows that T_d is bounded below and hence invertible. Thus by multiplying T on the right by T_d^{-1}, we may assume without loss of generality that T is of the form $I-S$ where $\| \, |S| \, \| < 1$. Hence each T_n is invertible; in fact, $T_n^{-1} = \sum_{k=0}^{+\infty} S_n^k$. Moreover,

$$\|\sup_n |T_n^{-1}| \, \| = \left\| \sup_n \left| \sum_{k=0}^{+\infty} S_n^k \right| \right\| \leqslant \left\| \sum_{k=0}^{+\infty} |S|^k \right\|$$
$$\leqslant \sum_{k=0}^{+\infty} \| \, |S| \, \|^k < +\infty.$$

Since T is order bounded, an application of Theorem 2 now gives the result.

The second method of extending diagonal dominance to other sequence spaces is based on the elementary observation (which we have already partially used in the proof of Corollary 4) that after multiplication by a diagonal operator a diagonally dominant operator on l_1 takes the form $I-S$, where $\|S\| \leqslant 1$. This suggests that it might be possible to factor operators that are close to the identity in some sense. This suspicion is confirmed by a result of Barkar and Gohberg [1, Theorem 5] which implies that if N is a nuclear operator on a classical Banach sequence space with $\|N\| \leqslant 1$ and if $I-N$ is invertible, then $I-N$ has an LU-factorization. We wish to extend this result to more general classes of operators than the nuclear operators. To this end we introduce some new norms on operators, whose form is suggested by a close examination of the proofs of Lemmas 3.3, 3.4, and 3.5 of [14]. Let $1 \leqslant q < +\infty$ and let T be an operator on a classical Banach sequence space such that $\sum_i \|Te_i\|^q < +\infty$. Then we define $|||T|||_q = (\sum_{i=1}^{+\infty} \|Te_i\|^q)^{1/q}$. For $q = +\infty$, the corresponding expression is $|||T|||_\infty = \sup_i \|Te_i\|$. We note that $|||T||| \geqslant \|T\|$ with equality if T operates on l_1. If T operates on l_2 then $|||T|||_2$ is the usual Hilbert–Schmidt norm [11, p. 214]. Finally, we remark that if T is a q-absolutely summing operator on l_p, where $1 < p < +\infty$ and $1/p + 1/q = 1$, then $|||T|||_q < +\infty$. This is because for q finite the q-absolutely summing norm of T is equal to $\sup\{(\sum_i \|Tx_i\|^q)^{1/q}\colon \sup\{(\sum_i |x^*(x_i)|^q)^{1/q}\colon \|x^*\| \leqslant 1\} \leqslant 1\}$, which always dominates $|||T|||_q$. Since a nuclear operator is q-absolutely summing for every $q \geqslant 1$, [11, p. 251], the next result may be viewed as a partial generalization of the aforementioned result of Barkar and Gohberg [1, Theorem 5].

THEOREM 5. *Let $1 < p \leqslant 2$ and $1/p + 1/q = 1$ and let $T = I - S$ be an invertible operator on l_p such that $|||S|||_q \leqslant 1$. Then T has an LU-factorization with L^{-1} and U^{-1} order bounded.*

Proof. We note first that Lemma 4.2 of [14] shows that the compressions T_n are uniformly invertible and so have LU-factorizations. In fact, $\|T_n^{-1}\| \leqslant 3\|T^{-1}\|^p$. (The proof states there for $p = 1$ can be modified to work for $p \geqslant 1$.) Now let $U_q(l_p)$ and $L_q(l_p)$ denote the spaces of upper triangular and strictly lower triangular operators on l_n^p endowed with the $|||\cdot|||_q$-norm. Define the operators $\hat{I}_n - P_{S_n}$ and $\hat{I}_n - Q_{S_n}$ on $U_q(l_p^n)$ and $L_q(l_p^n)$, respectively, by

$$[\hat{I}_n - P_{S_n}][A] = A - (S_n A)_+ \qquad \text{for all } A \text{ in } U_q(l_p^n)$$

and

$$[\hat{I}_n - Q_{S_n}][A] = A - (AS_n)_- \qquad \text{for all } A \text{ in } L_q(l_p^n).$$

Here $(S_n A)_+$ is the upper triangular part of $S_n A$ and $(AS_n)_-$ is the strictly lower triangular part of AS_n.

It suffices to show that these operators are uniformly invertible. For once this is done, we have that if $L_n U_n$ is the LU-factorization of T_n, then $U_n^{-1} - \mathbf{I}_n = [\hat{I}_n - P_{S_n}]^{-1}[S_n]_+$ and $L_n^{-1} - \mathbf{I}_n = [\hat{I}_n - Q_{S_n}]^{-1}[S_n]_-$. Consequently,

$$\begin{aligned} \|\sup_n |T_n^{-1}|\,\| &= \|\sup_n |U_n^{-1} L_n^{-1}|\,\| \leqslant \sup_n |||U_n^{-1}|||\ |||L_n^{-1}||| \\ &\leqslant \sup_n (1 + |||U_n^{-1} - \mathbf{I}_n|||_q)(1 + |||L_n^{-1} - \mathbf{I}_n|||_q) < +\infty \end{aligned}$$

and so Theorem 2 gives the result.

We now establish the *uniform* invertibility of $\hat{I}_n - P_{S_n}$ and $\hat{I}_n - Q_{S_n}$. It is fairly easy to give arguments that show that $\hat{I} - P_{S_n}$ and $\hat{I}_n - Q_{S_n}$ are invertible. For example, suppose that there exists a $U_n \in U_q(l_p^n)$ so that $\|\,|U_n|\,\| = \hat{1}$ and $(I_n - P_S)(U_n) = 0$. Then $U_n e_i = P_i S_n U_n e_i$ for all i. Now since U_n is upper triangular, $P_i U_n e_i = U_n e_i$ for all i. Thus $U_n e_i = P_i S_n P_i U_n e_i = S_i U_n e_i$ and so $(I_i - S_i)(U_n e_i) = 0$, which contradicts the invertibility of $I_i - S_i$. Hence, $I_n - P_{S_n}$ is 1–1 and so invertible. A similar but more complicated argument shows that $\hat{I}_n - Q_{S_n}$ is 1–1 and hence invertible. The difficulty with these arguments is that they do not relate the norms $\|(\hat{I}_n - Q_{S_n})^{-1}\|$ and $\|(\hat{I}_n - P_{S_n})^{-1}\|$ with $\|(I_n - S_n)^{-1}\|$; hence, they must be modified in order to establish the uniform invertibility of $\hat{I}_n - Q_{S_n}$ and $\hat{I}_n - P_{S_n}$. To do this requires a series of technical lemmas which are based on Lemmas 3.3., 3.4, and 3.5 of [14] where the case $p = 1$ is treated. In a sense, $p = 1$ is the most difficult case. This is because for operators T on l_1, $|||T|||_\infty = \|T\|$, while if T is an operator on l_p, $1 < p < +\infty$, then $|||T|||_q \geqslant \|T\|$, where $1/p + 1/q = 1$. Thus, the hypothesis that $|||T|||_q \leqslant 1$ is much more stringent for $p > 1$ than $p = 1$. We begin by establishing the uniform invertibility of $\hat{I}_n - P_{S_n}$.

LEMMA 6. *Let* $1 < p \leqslant 2$ *and let* $1/p + 1/q = 1$. *If* $I - S$ *is invertible on* l_p

and $|||S|||_q \leqslant 1$, *then, for each,* n $\hat{I}_n - P_{S_n}$ *is invertible on* $U_q(l_p^n)$ *and* $\sup\|(\hat{I}_n - P_{S_n})^{-1}\| < +\infty$.

Proof. We will make use of the elementary fact that if $1 < p \leqslant 2$ and $1/p + 1/q = 1$, then for any two real numbers y and z such that $0 \leqslant y, z \leqslant 1$, we have that $(y^q + z^q) \leqslant (y^p + z^p)^{q/p}$. Now let $U_n \in U_q(l_p^n)$ with $|||U_n|||_q = 1$. Then for each i, $(\|(I - P_i)\, SU_n e_i\|^q + \|P_i SU_n e_i\|^q) \leqslant (\|(I - P_i)\, SU_n e_i\|^p + \|P_i SU_n e_i\|^p)^{q/p} = (\|SU_n e_i\|^p)^{q/p} = \|SU_n e_i\|^q$. Suppose that $\hat{I}_n - P_{S_n}$ is not invertible. Then for each $0 < \varepsilon < 1$ there is a $U_n \varepsilon U_q(l_p^n)$ of norm 1 such that $|||(\hat{I}_n - P_{S_n})(U_n)||| \leqslant \varepsilon$. Then $(\sum \|(U_n - (S_n U_n)_+)\, e_i\|^q)^{1/q} \leqslant \varepsilon$, so $(\sum \|(U_n - P_i S_n U_n)\, e_i\|^q)^{1/q} \leqslant \varepsilon$. Since $(\sum \|U_n e_i\|^q)^{1/q} = 1$, it follows that $(\sum \|P_i S_n U_n e_i\|^q)^{1/q} > 1 - \varepsilon$. Consequently, $\sum \|(I_n - P_i)\, S_n U_n e_i\|^q \leqslant \sum \|S_n U_n e_i\|^q - \sum \|P_i S_n U_n e_i\|^q \leqslant 1 - (1 - \varepsilon)^q < q\varepsilon$ and so $(\sum \|(I_n - S_n)\, U_n e_i\|^q)^{1/q} \leqslant (\sum \|(I_n - P_i)(SU_n e_i\|^q)^{1/q} + (\sum \|(I_n - P_i S_n)\, U_n e_i\|^q)^{1/q} \leqslant (q)^{1/q} + \varepsilon < (q^{1/q} + 1)(\varepsilon^{1/q})$. Hence, $1 = (\sum \|U_n e_i\|^q)^{1/q} \leqslant \|(I_n - S_n)^{-1}\|\, (\sum \|(I_n - S_n)\, U_n e_i\|^q)^{1/q} \leqslant \|(I_n - S_n)^{-1}\| [(q)^{1/q} + 1]\, \varepsilon^{1/q}$, a contradiction for ε close to zero. It follows that $\hat{I}_n - P_{S_n}$ is bounded below and hence invertible. Moreover, if $\|(\hat{I}_n - P_{S_n})^{-1}\| \geqslant 1$ then $1 \leqslant \|(I_n - S_n)^{-1}\| (q^{1/q} + 1) \| (\hat{I}_n - P_{S_n})^{-1}\|^{-1/q}$ and finally $\|(\hat{I}_n - P_{S_n})^{-1}\| \leqslant (q^{1/q} + 1)^q \|(I_n - S_n)^{-1}\|^q$. Thus $\sup_n \|(\hat{I}_n - P_{S_n})^{-1}\| \leqslant (q^{1/q} + 1)^q \|(I - S)^{-1}\|^q + 1 < +\infty$, as desired.

To establish the uniform invertibility of $\hat{\mathbf{I}}_n - Q_{S_n}$ we need a preliminary lemma.

LEMMA 7. *Let* $1 < p \leqslant 2$ *and* $1/p + 1/q = 1$. *Let* S, V *be in* $B(l_p^n)$ *with* $|||S|||_q \leqslant 1$ *and* $|||V|||_q = 1$. *If there exists a* $1 > \delta > 0$ *and* $x_i \in l_p^n$ *such that* $(\sum_i \|x_i\|^q)^{1/q} \leqslant 1$, $(\sum_i \|Vx_i\|^q)^{1/q} > 1 - \delta$ *and* $(\sum_j \|Ve_j - (VS)_- e_j\|^q)^{1/q} < \delta$, *then* $(\sum_i \|Vx_i - VSx_i\|^q)^{1/q} < 5\delta$.

Proof. For any set of positive integers J, let $P_J x = \sum_{j \in J} \langle x, e_j \rangle e_j$, and $P_{\bar{J}} x = \sum_{j \notin J} \langle x, e_j \rangle e_j$. Since $|||V|||_q = 1$, there exists a set of positive integers J such that $(\sum_{j \in J} \|Ve_j\|^q)^{1/q} > 1 - \delta$ and $(\sum_i \|P_{\bar{J}} x_i\|^q)^{1/q} < \delta$. Consequently, $(\sum_{j \in J} \|(VS)_- e_j\|^q)^{1/q} > 1 - \delta - \delta = 1 - 2\delta$ and so $(\sum_{j \in J} \|(VS)_+ e_j\|^q)^{1/q} < 2\delta$. It follows that

$$\left(\sum_i \|VSx_i - (VS)_- x_i\|^q\right)^{1/q} = \left(\sum_i \left\|\sum_j \langle x_i, e_j \rangle (VS)_+ e_j\right\|^q\right)^{1/q}$$

$$\leqslant \left(\sum_i \left\|\sum_{j \in J} \langle x_i, e_j \rangle (VS)_+ e_j\right\|^q\right)^{1/q}$$

$$+ \left(\sum_i \left\|\sum_{j \notin J} \langle x_i, e_j \rangle (VS)_+ e_j\right\|^q\right)^{1/q}$$

$$\leqslant \left(\sum_i \|P_J x_i\|^q\right)^{1/q} \left(\sum_{j\in J} \|(VS)_+ e_j\|^q\right)^{1/q}$$

$$+ \left(\sum_i \|P_{\bar{J}} x_i\|^q\right)^{1/q} \left(\sum_{j\notin J} \|(VS)_+ e_j\|^q\right)^{1/q}$$

$$\leqslant (1)(2\delta) + (2\delta)(1) = 4\delta.$$

Finally, we have that

$$\left(\sum_i \|Vx_i - VSx_i\|^q\right)^{1/q}$$

$$\leqslant \left(\sum_i \|[V-(VS)_-]\, x_i\|^q\right)^{1/q} + \left(\sum_i \|[(VS)_- - VS]\, x_i\|^q\right)^{1/q}$$

$$\leqslant \left(\sum_i \|x_i\|^q\right)^{1/q} \left(\sum_j \|[V-(VS)_-]\, e_j\|^q\right)^{1/q} + 4\delta \leqslant \delta + 4\delta \leqslant 5\delta.$$

This completes the proof.

We can now proceed to establish the uniform invertibility of $\hat{I}_n - Q_{S_n}$.

LEMMA 8. *Let* $1 < p \leqslant 2$ *and* $1/p + 1/q = 1$. *If* $I - S$ *is invertible on* l_p *and* $|||S|||_q \leqslant 1$, *then for each* n, $I_n - S_n$ *is invertible on* $L_q(l_p^n)$ *and* $\sup_n \|(\hat{I}_n - Q_{S_n})^{-1}\| < +\infty$.

Proof. If not, then for each $\varepsilon > 0$ (in particular, for $0 < \varepsilon < 1$), there exists an n and a $V_n \varepsilon L_q(l_p^n)$ such that $|||V|||_q = 1$ and $|||V_n - (V_n S_n)_-|||_q < \varepsilon$. Since $(\sum_i \|V_n e_i\|^q)^{1/q} = 1$ and $(\sum_i \|V_n e_i - (V_n S_n)_- e_i\|^q)^{1/q} < \varepsilon$, it follows that $(\sum_i \|(V_n S_n)_- e_i\|^q)^{1/q} > 1 - \varepsilon$. Since $(\sum_i \|V_n S_n e_i\|^q)^{1/q} \leqslant 1$, we have that $(\sum_i \|(V_n S_n)_+ e_i\|^q)^{1/q} < \varepsilon$. Hence $(\sum_i \|V_n e_i - V_n S_n e_i\|^q)^{1/q} < (\sum_i \|V_e e_i - (V_n S_n) e_i\|^q)^{1/q} + (\sum_i \|(V_n S_n) e_i\|^q)^{1/q} < \varepsilon + \varepsilon = 2\varepsilon$. It follows that $(\sum_i \|V_n S_n e_i\|^q)^{1/q} > 1 - 2\varepsilon$. If $2\varepsilon < 1$ we may apply Lemma 7 with $\delta = 2\varepsilon$ and $x_i = S_n e_i$ to conclude that $(\sum_i \|V_n S_n e_i - V_n S_n^2 z_i\|^q)^{1/q} < 5(2\varepsilon)$. Thus $(\sum_i \|V_n S_n^2 e_i\|^q)^{1/q} > 1 - 5(2\varepsilon)$. If $5(2\varepsilon) < 1$, we may apply Lemma 7 again with $x_i = S_n^2 e_i$ and $\delta = 5(2\varepsilon)$ to conclude that $(\sum_i \|V_n S_n^2 e_i - V_n S_n^3 e_i\|^q)^{1/q} < 5^2(2\varepsilon)$. In general, we obtain that for each nonnegative integer j that $(\sum_i \|V_n S_n^j e_i - V_n S_n^{j+1} e_i\|^q)^{1/q} < 5^j(2\varepsilon)$ and $(\sum_i \|V_n S_n^{j+1} e_i\|^q)^{1/q} > 1 - 5^j(2\varepsilon)$ provided $5^j(2\varepsilon) < 1$. Now for an integer m such that $5^m(2\varepsilon) < 1$, define $L_m = (S_n + S_n^2 + S_n^3 + \cdots + S_n^m)/m$. Then

$$|||L_m|||_q \geqslant |||V_n L_m|||_q = (1/m)|||V_n S_n + V_n S_n^2 + \cdots + V_n S_n^m|||_q$$

$$\geqslant |||V_n S_n|||_q - (1/m) \sum_{k=1}^{m} |||V_n S_n^k - V_n S_n|||_q$$

$$\geqslant 1-2\varepsilon-(1/m)\sum_{k=1}^{m}\sum_{j=0}^{k-1}|||V_nS_n^j-V_nS_n^{j+1}|||_q$$

$$\geqslant 1-2\varepsilon-(1/m)\sum_{k=1}^{m}\sum_{j=0}^{k-1}5^j 2\varepsilon.$$

On the other hand, $(I_n-S_n)L_m=(S_n-S_n^{m+1})/m$ and so

$$|||L_m|||_q\leqslant\|(I_n-S_n)^{-1}\|\ |||(I_n-S_n)L_m|||_q$$
$$\leqslant\|(I_n-S_n)^{-1}\|2/m\leqslant 4\|(I-S)^{-1}\|/m.$$

But then $1-2-1/m\sum_{k=1}^{m}\sum_{j=0}^{k-1}5^j(2\varepsilon)\leqslant 4/m\|(I-S)^{-1}\|$, which is impossible for m large and ε close to zero. This contradiction completes the proof of this lemma, and so the proof of Theorem 5 is complete.

As a small illustration of the utility of LU-factorizations we offer the following modification of a result of Shinbrot [12]. Here $Q_n=I-P_n$ is the complementary projection to P_n.

THEOREM 9. *Let T be an operator on a classical Banach sequence space X. If T has an LU-factorization, then for each n and $y\in X$, the equation $Q_nx+TP_nx=y$ has a unique solution given by*

$$P_nx=U_n^{-1}P_nL^{-1}y$$

and (*)

$$Q_nx=LQ_nL^{-1}y.$$

Proof. Suppose that x is a solution of $Q_nx+TP_nx=y$. Then $Q_nx+LUP_nx=y$. Applying P_nL^{-1} to both sides, we obtain $P_nL^{-1}Q_nx+P_nUP_nx=P_nL^{-1}y$. But $P_nL^{-1}Q_n=0$ since L^{-1} is lower triangular. Hence $U_nx=P_nL^{-1}x$. Since U_n is invertible (relative to X_n), we have that $P_nx=U_n^{-1}P_nL^{-1}x$. Since U is upper triangular, $L^{-1}Q_nx=L^{-1}y-UP_nx=L^{-1}y-P_nUP_nx=L^{-1}y-U_nU_n^{-1}P_nL^{-1}y=Q_nL^{-1}y$ and so $Q_nx=LQ_nL^{-1}y$. Hence, the solution is unique. On the other hand, it is easily checked that (*) defines a solution of the operator equation.

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